1 2 9 0

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Nome Completo do Autor

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Subtítulo

VOLUME 1

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pelo/a Professor/a Doutor/a <nome completo do orientador> e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra e ao Departamento de Matemática da Faculdade de Ciências da Universidade do Porto.

Mês de 20XX

Title of PhD thesis

Author's Name

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Programa Interuniversitário de Doutoramento em Matemática

PhD Thesis | Tese de Doutoramento

July 2019

Acknowledgements

I would like to acknowledge ...

Abstract

This is where you write your abstract in English.

Resumo

Aqui escreve o Resumo em Português.

Table of contents

List of figures

List of tables

Chapter 1

Introduction

1.1 First section of the first chapter

And now I begin my first chapter here . . .

And now to cite some people **? ?**]

How to index a name:

A *LaTeX class file* is a file, which holds style information for a particular LaTeX.

How to start a list of Nomenclature and Notation (OPTIONAL): ...

1.1.1 First subsection in the first section

. . . and some more

1.1.2 Second subsection in the first section

. . . and some more . . .

First subsub section in the second subsection

. . . and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it . . .

1.1.3 Third subsection in the first section

. . . and some more . . .

First subsub section in the third subsection

. . . and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it and some more and some more and some more and some more and some more and some more and some more . . .

Second subsub section in the third subsection

. . . and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it . . .

1.2 Second section of the first chapter

and here I write more . . .

1.3 The layout of formal tables

This section has been modified from "Publication quality tables in LaTeX*" by Simon Fear.

The layout of a table has been established over centuries of experience and should only be altered in extraordinary circumstances.

When formatting a table, remember two simple guidelines at all times (see Table 1.4):

1. Never, ever use vertical rules (lines).

2. Never use double rules.

These guidelines may seem extreme but I have never found a good argument in favour of breaking them. For example, if you feel that the information in the left half of a table is so different from that on the right that it needs to be separated by a vertical line, then you should use two tables instead. Not everyone follows the second guideline:

There are three further guidelines worth mentioning here as they are generally not known outside the circle of professional typesetters and subeditors:

3. Put the units in the column heading (not in the body of the table).

4. Always precede a decimal point by a digit; thus 0.1 *not* just .1.

5. Do not use 'ditto' signs or any other such convention to repeat a previous value. In many circumstances a blank will serve just as well. If it won't, then repeat the value.

A frequently seen mistake is to use '\begin{center}' . . . '\end{center}' inside a figure or table environment. This center environment can cause additional vertical space. If you want to avoid that just use '\centering'

These guidelines may seem extreme but I have never found a good argument in favour of breaking them. For example, if you feel that the information in the left half of a table is so different from that on the right that it needs to be separated by a vertical line, then you should use two tables instead. Not everyone follows the second guideline:

There are three further guidelines worth mentioning here as they are generally not known outside the circle of professional typesetters and subeditors:

Table 1.1 A badly formatted table

	Species I		Species II	
Dental measurement	mean	SD	mean	SD
I1MD	6.23	0.91	5.2	0.7
I1LL	7.48	0.56	8.7	0.71
I2MD	3.99	0.63	4.22	0.54
I2LL	6.81	0.02	6.66	0.01
CMD	13.47	0.09	10.55	0.05
CBL	11.88	0.05	13.11	0.04

Table 1.2 A nice looking table

Dental measurement	Species I		Species II	
	mean	SD	mean	SD
I1MD	6.23	0.91	5.2	0.7
I1LL	7.48	0.56	8.7	0.71
I2MD	3.99	0.63	4.22	0.54
I2LL	6.81	0.02	6.66	0.01
CMD	13.47	0.09	10.55	0.05
CBL	11.88	0.05	13.11	0.04

Table 1.3 Even better looking table using booktabs

Dental measurement	Species I		Species II	
	mean	SD	mean	SD
I1MD	6.23	0.91	5.2	0.7
I1LL	7.48	0.56	8.7	0.71
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CMD	13.47	0.09	10.55	0.05
CBL	11.88	0.05	13.11	0.04

Table 1.4 Characterizations of normal and extremally disconnected spaces

Space X	NORMAL	EXTREMALLY DISCONNECTED
Urysohn's separation type lemma	Every two disjoint CLOSED subsets of X are completely separated (Urysohn 1925).	Every two disjoint OPEN subsets of X are completely separated (Gillman & Jerison 1960).
Tietze's extension type theorem	Each CLOSED subset of X is C^*-embedded (Tietze 1915).	Each OPEN subset of X is C^*-embedded (Gillman & Jerison 1960).
Katětov-Tong insertion type theorem	For every UPPER semicontinuous real function f and LOWER semicontinuous real function g satisfying $f \leq g$, there exists a continuous real function h such that $f \leq h \leq g$ (Katětov 1951, Tong 1952).	For every LOWER semicontinuous real function f and UPPER semicontinuous real function g satisfying $f \leq g$, there exists a continuous real function h such that $f \leq h \leq g$ (Stone 1949, Lane 1975).
Hausdorff mapping invariance type theorem	The image of X under any CLOSED continuous map is NORMAL (Hausdorff 1935).	The image of X under any OPEN continuous map is EXTREMALLY DISCONNECTED.

1.4 Next section

1.5 Next section

1.6 Next section

1.7 Next section

1.8 Next section

1.9 Next section

Chapter 2

My second chapter

2.1 Reasonably long section title

I'm going to randomly include a picture in Figure 2.1.

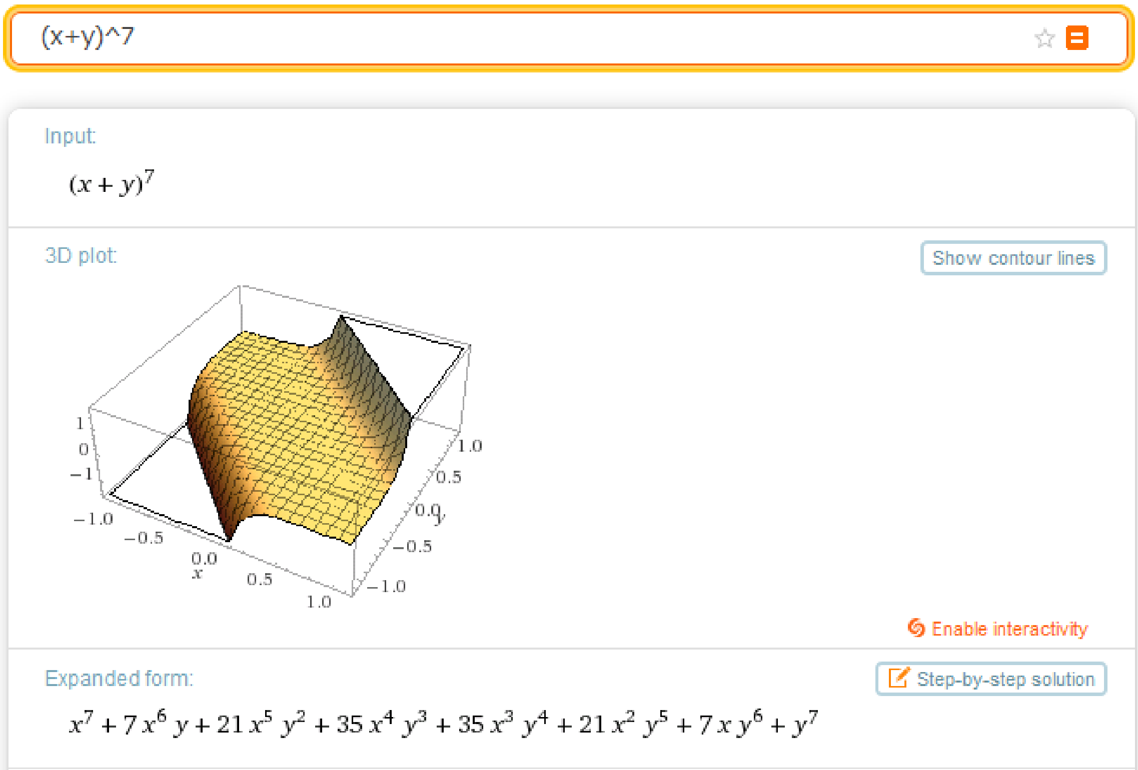


Fig. 2.1 This is just a long figure caption

Enumeration

1. The first topic is dull

2. The second topic is duller
    (a) The first subtopic is silly
    (b) The second subtopic is stupid
3. The third topic is the dullest

itemize

- The first topic is dull
- The second topic is duller
    - The first subtopic is silly
    - The second subtopic is stupid
- The third topic is the dullest

description

The first topic is dull

The second topic is duller

The first subtopic is silly

The second subtopic is stupid

The third topic is the dullest

2.2 Second section

Galois was born on 25 October 1811 to Nicolas-Gabriel Galois and Adélaïde-Marie (born Demante). His father was a Republican and was head of Bourg-la-Reine's liberal party. He became mayor of the village after Louis XVIII returned to the throne in 1814. His mother, the daughter of a jurist, was a fluent reader of Latin and classical literature and was responsible for her son's education for his first twelve years. At the age of 10, Galois was offered a place at the college of Reims, but his mother preferred to keep him at home.

In October 1823, he entered the Lycée Louis-le-Grand, and despite some turmoil in the school at the beginning of the term (when about a hundred students were expelled), Galois managed to perform well for the first two years, obtaining the first prize in Latin. He soon became bored with his studies and, at the age of 14, he began to take a serious interest in mathematics.

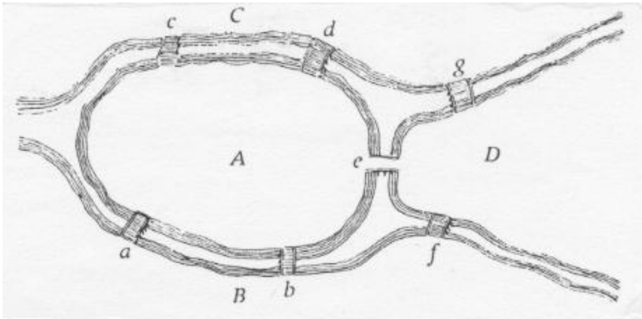
He found a copy of Adrien Marie Legendre's *Éléments de Géométrie*, which it is said that he read "like a novel" and mastered at the first reading. At 15, he was reading the original papers of Joseph Louis Lagrange, such as the landmark *Réflexions sur la résolution algébrique des équations* which likely motivated his later work on equation theory, and *Leçons sur le calcul des fonctions*, work intended for professional mathematicians, yet his classwork remained uninspired, and his teachers accused him of affecting ambition and originality in a negative way.[1].

While many mathematicians before Galois gave consideration to what are now known as groups, it was Galois who was the first to use the word group (in French *groupe*) in a sense close to the technical sense that is understood today, making him among the founders of the branch of algebra known as group theory. He developed the concept that is today known as a normal subgroup. He called the decomposition of a group into its left and right cosets a proper decomposition if the left and right cosets coincide, which is what today is known as a normal subgroup. He also introduced the concept of a finite field (also known as a Galois field in his honor), in essentially the same form as it is understood today.

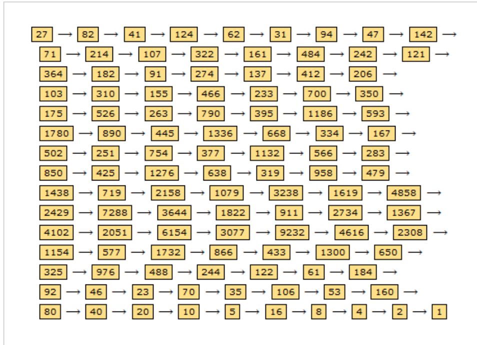
In his last letter to Chevalier and attached manuscripts, the second of three, he made basic studies of linear groups over finite fields:

- He constructed the general linear group over a prime field, $GL(\nu, p)$ and computed its order, in studying the Galois group of the general equation of degree p^ν.
- He constructed the projective special linear group $PSL(2, p)$. Galois constructed them as fractional linear transforms, and observed that they were simple except if p was 2 or 3. These were the second family of finite simple groups, after the alternating groups.
- He noted the exceptional fact that $PSL(2, p)$ is simple and acts on p points if and only if p is 5, 7, or 11.

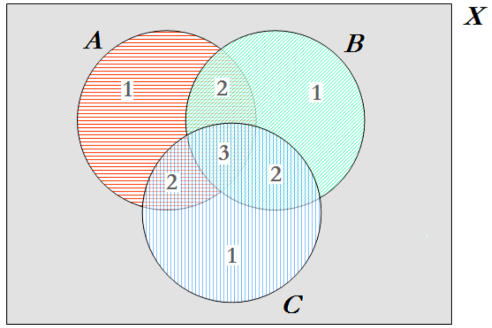
[1] My footnote goes blah blah blah! ...



(a) 7 Bridges



(b) Collatz



(c) Venn diagram

Fig. 2.2 Best Pictures

Subplots

I can cite AAA (see Fig. 2.2b) and BBB (Fig. 2.2c) or I can cite the whole figure as Fig. 2.2

2.3 Third section

2.4 Fourth section

2.5 Hidden section

Chapter 3

My third chapter

3.1 Title with math σ

The well known Pythagorean theorem $x^2 + y^2 = z^2$ was proved to be invalid for other exponents. Meaning the next equation has no integer solutions: $x^n + y^n = z^n$.

The binomial coefficient is defined by the next expression:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

And of course this command can be included in the normal text flow $\binom{n}{k}$. Limit $\lim_{x \to \infty} f(x)$ inside text.

$$\lim_{x \to \infty} f(x)$$

The most famous equation in the world: $E^2 = (m_0c^2)^2 + (pc)^2$, which is known as the **energy-mass-momentum** relation as an in-line equation.

$$CIF: \quad F_0^j(a) = \frac{1}{2\pi\imath} \oint_\gamma \frac{F_0^j(z)}{z-a} dz \tag{3.1}$$

Integral $\int_a^b x^2 dx$ inside text.

$$\iiint_V \mu(u,v,w)\, du\, dv\, dw \tag{3.2}$$

3.2 Preliminaries I. Free constructions

We will work with point-free real numbers as they are usually described in literature, that is, by generators subject to relations. Since the free generators come from a set that is in fact a meet-semilattice (while its elements are used in the free construction simply as elements of a set) we think that it may be useful for the reader to compare the free frames over sets with free frames over semilattices.

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3.2.1 Free semilattice with 1.

For a set X define $F(X) = \{A \subseteq X \mid A \text{ finite}\}$ ordered by $\leq = \supseteq$ so that we have the meet $A \wedge B = A \cup B$. Denote by β_X the mapping

$$\beta_X = (x \mapsto \{x\})\colon X \to F(X).$$

Then we have for each meet-semilattice S with 1 and each mapping $f\colon X \to S$ precisely one meet-semilattice homomorphism $\overline{f}\colon F(X) \to S$ such that $\overline{f}\beta_X = f$ and $\overline{f}(\emptyset) = 1$, namely the homomorphism defined by $\overline{f}(A) = \bigwedge_{x \in A} f(x)$.

3.2.2 Free frame generated by a semilattice with 1.

For a meet-semilattice S with 1 set $\mathfrak{D}(S) = \{U \subseteq S \mid \downarrow U = U \neq \emptyset\}$. $\mathfrak{D}(S)$ is a frame with unions for joins and intersections for meets and if we denote by α_S the mapping

$$\alpha_S = (s \mapsto \downarrow s)\colon S \to \mathfrak{D}(S)$$

we have a meet-semilattice homomorphism such that for each frame L and each meet-semilattice homomorphism $h\colon S \to L$ there is precisely one frame homomorphism $\widehat{h}\colon \mathfrak{D}(S) \to L$ such that $\widehat{h}\alpha_S = h$, namely that defined by $\widehat{h}(U) = \bigvee_{s \in U} h(s)$.

The free frame over a set can be now obtained combining F and $\mathfrak{D}$, that is, as $\mathfrak{D}F(X)$.

3.2.3 Free frames over a set and over a meet-semilattice compared.

Now suppose we have a construction of a frame based on a set which is endowed by a meet-semilattice structure. We will compare the free constructions as over the carrier $|S|$ and the one based directly on the semilattice S.

We will use the standard factorization procedure as e.g. in [**?** , III.11]. On $\mathfrak{D}F(|S|)$ define a relation

$$M = \{(\downarrow A, \downarrow B) \mid \bigwedge A = \bigwedge B \text{ in } S\}$$

and write $\kappa\colon \mathfrak{D}F(|S|) \to \mathfrak{D}F(|S|)/M$ for the quotient map. Consider the following diagram:

$$\begin{tikzcd}
{|S|} \arrow[r, "\beta_{|S|}"] \arrow[d, "\mathrm{id}"'] & F(|S|) \arrow[r, "\alpha_{F(|S|)}"] \arrow[d, "h=\overline{\mathrm{id}}"] & \mathfrak{D}F(|S|) \arrow[dl, "\widehat{h}"'] \arrow[d, "\kappa"] \\
S \arrow[r, "\alpha_S"] \arrow[rr, bend right, "f"'] & \mathfrak{D}(S) \arrow[r, bend right=10, "\psi"'] & \mathfrak{D}F(|S|)/M \arrow[l, bend right=10, "\phi"']
\end{tikzcd}$$

Since $\widehat{h}$ obviously respects the relation M we have a frame homomorphism ϕ such that $\phi\kappa = \widehat{h}$. Further, define a mapping

$$f\colon S \to \mathfrak{D}F(|S|)/M$$

by setting $f(s) = \kappa(\downarrow\{s\})$. By the definition of M, f is a meet-semilattice homomorphism and hence there is a frame homomorphism ψ such that $\psi\alpha_S = f$. Now we have

$$\begin{aligned}
&\phi\psi(\downarrow s) = \phi\psi\alpha_S(s) = \phi f(s) = \phi\kappa(\downarrow\{s\}) = \widehat{h}(\downarrow\{s\}) = \\
&\qquad = \widehat{h}\alpha_{F(|S|)}(\{s\}) = h(\{s\}) = h\beta_{|S|}(s) = \alpha_S(s) = \downarrow s \quad \text{and} \\
&\psi\phi(\kappa(\downarrow\{s\})) = \psi\widehat{h}(\downarrow\{s\}) = \psi\widehat{h}\alpha_{F(|S|)}(\{s\}) = \psi\alpha_S(s) = f(s) = \kappa(\downarrow\{s\})
\end{aligned}$$

so that $\phi\psi$ and $\psi\phi$ are identical on systems of generators and hence ϕ and ψ are mutually inverse homomorphisms.

Thus, if we represent a construction based on factorizing $\mathfrak{D}(S)$ identifying pairs from a relation R as a free construction on $|S|$ we only have to consider the relation $R \cup M$ instead of R, with the M as above.

3.3 Where does it come from?

Thus, if we represent a construction based on factorizing $\mathfrak{D}(S)$ identifying pairs from a relation R as a free construction on $|S|$ we only have to consider the relation $R \cup M$ instead of R, with the M as above.

The frame $\mathfrak{L}(\mathbb{R})$ is the completion of $\Omega(\mathbb{Q})$ (both taken with the uniformity derived from the respective metric uniformities), where the completion homomorphism $\gamma\colon \mathfrak{L}(\mathbb{R}) \to \Omega(\mathbb{Q})$ is given by $(p,q) \mapsto]p,q[= \{x \in \mathbb{Q} \mid p < x < q\}$. Then $\mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R})$ (n summands) is the completion of $\Omega(\mathbb{Q}^n)$ with the completion map $\overline{\gamma}$ given by the coproduct diagram

$$\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[r, "\iota_i"] \arrow[d, "\gamma"'] & \mathfrak{L}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{L}(\mathbb{R}) \arrow[d, dotted, "\overline{\gamma}"] \\
\Omega(\mathbb{Q}) \arrow[r, "\Omega(p_i)"'] & \Omega(\mathbb{Q}^n)
\end{tikzcd}$$

(where the p_i, $i = 1,\dots,n$, are the projections $\mathbb{Q}^n \to \mathbb{Q}$).

The frame of reals, $\mathfrak{L}(\mathbb{R})$ ("point-free real numbers"), was originally introduced by Joyal in an unpublished manuscript and thoroughly studied by Banaschewski in [**?**] (see also Johnstone [**?**]). As one might expect, it was not defined as the lattice $\Omega(\mathbb{R})$ of open sets in the standard real line $\mathbb{R}$ but as a primarily algebraic entity, the free frame generated by pairs of rational numbers (which one can intuitively view as rational intervals) factorized by natural relations (see 2.3 below). Under the Axiom of Choice, $\mathfrak{L}(\mathbb{R})$ is indeed isomorphic with $\Omega(\mathbb{R})$, but the point is to have the frame of point-free reals as a frame in its own right and to be able to avoid choice whenever possible (it should be noted that one can prove in a choice-free way for instance that $\mathfrak{L}(\mathbb{R})$ is the completion of the frame of rationals or that it is continuous, that is, locally compact, see [**?**]).

Once one has the frame of real numbers, one can also represent continuous real functions on a general frame L, namely as frame homomorphisms $h\colon \mathfrak{L}(\mathbb{R}) \to L$. This was originally done by Banaschewski. However, the classical theory of real functions, not necessarily continuous, calls for a point-free counterpart as well. An appropriate definition was presented in [**?**]. A classical (general) real function on a space $(X,\Omega(X))$ is a continuous real function on the discrete space $(X,\mathfrak{P}(X))$. The lattice $\mathfrak{P}(X)$ of all subsets of X has a natural counterpart in $\mathscr{S}(L)^{\mathrm{op}}$ where $\mathscr{S}(L)$ is the co-frame of all sublocales of L. Hence, a (general) real function on L can be represented as a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathscr{S}(L)^{\mathrm{op}}$.

The present paper is inspired by [**?**]. Using extensively the technique of sublocales, we present a survey of some facts on point-free real functions. Most of the results are not new; the originality is essentially in the presentation. Our main goal is to show how zero sets may be considered in the localic setting (as *zero sublocales*) and then how several important notions and results about real functions may be rewritten and directly proved using this tool.

After some necessary preliminaries we introduce the point-free real functions and prove a few facts, in particular some results concerning images and preimages of sublocales are discussed. Then, semicontinuous functions and their relation with the continuous ones are mentioned. In the following section, point-free algebraic operations on $\mathfrak{L}(\mathbb{R})$ are studied, with special attention paid to the addition, multiplication, maximum and minimum. Next we turn to cozero and zero sublocales. The concept of *cozero element* is a well-known standard topic and its sublocale counterpart is straightforward, but there are no reasonable *zero elements* while in the context of sublocales we obtain a sensible notion. This approach allows to formulate the basics of the theory in a way very much parallel to the classical book of Gillman and Jerison [**?**]. We illustrate this in a miscellany of topics.

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This approach allows to formulate the basics of the theory in a way very much parallel to the classical book of Gillman and Jerison [**?**]. We illustrate this in a miscellany of topics.

3.4 Next section

3.5 Next section

3.6 Next section

Chapter 4

Conclusion

We end with some final comments . . .

Appendix A

More information

Carl Friedrich Gauss

Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855) was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy, matrix theory, and optics.

Sometimes referred to as the Princeps mathematicorum (Latin, "the Prince of Mathematicians" or "the foremost of mathematicians") and "greatest mathematician since antiquity", Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians.

Gauss was a child prodigy. There are many anecdotes about his precocity while a toddler, and he made his first ground-breaking mathematical discoveries while still a teenager. He completed Disquisitiones Arithmeticae, his magnum opus, in 1798 at the age of 21, though it was not published until 1801. This work was fundamental in consolidating number theory as a discipline and has shaped the field to the present day.

Gauss's intellectual abilities attracted the attention of the Duke of Brunswick, who sent him to the Collegium Carolinum (now Braunschweig University of Technology), which he attended from 1792 to 1795, and to the University of Göttingen from 1795 to 1798. While at university, Gauss independently rediscovered several important theorems; his breakthrough occurred in 1796 when he showed that any regular polygon with a number of sides which is a Fermat prime (and, consequently, those polygons with any number of sides which is the product of distinct Fermat primes and a power of 2) can be constructed by compass and straightedge. This was a major discovery in an important field of mathematics; construction problems had occupied mathematicians since the days of the Ancient Greeks, and the discovery ultimately led Gauss to choose mathematics instead of philology as a career. Gauss was so pleased by this result that he requested that a regular heptadecagon be inscribed on his tombstone. The stonemason declined, stating that the difficult construction would essentially look like a circle.

The year 1796 was most productive for both Gauss and number theory. He discovered a construction of the heptadecagon on 30 March. He further advanced modular arithmetic, greatly simplifying manipulations in number theory. On 8 April he became the first to prove the quadratic reciprocity law. This remarkably general law allows mathematicians to determine the solvability of any quadratic equation in modular arithmetic. The prime number theorem, conjectured on 31 May, gives a good understanding of how the prime numbers are distributed among the integers.

Another section

Subsection

Subsubsection

...